

# Analysis of Double Covers of Factor Graphs

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**Abstract**—Many quantities of interest in communications, signal processing, artificial intelligence, and other areas can be expressed as the partition sum of some factor graph. Although the exact calculation of the partition sum is in many cases intractable, it can often be approximated rather well by the Bethe partition sum. In earlier work, we have shown that graph covers are a useful tool for expressing and analyzing the Bethe approximation. In this paper, we present a novel technique for analyzing double covers, a technique which ultimately leads to a deeper understanding of the Bethe approximation.

## I. INTRODUCTION

Consider a normal factor graph (NFG)  $N$  (see [1]–[3]). Its partition sum is defined to be

$$Z(N) \triangleq \sum_{\mathbf{a} \in \mathcal{A}} g(\mathbf{a}), \quad (1)$$

where the sum is over all configurations of  $N$  and where  $g$  is the global function of  $N$ . For example, the constrained capacity of a storage system can be expressed as the partition sum of a suitably formulated NFG (see, e.g., [4], [5]).

Because in many cases of interest the quantity  $Z(N)$  is intractable, people have come up with various techniques for efficiently approximating  $Z(N)$ . For NFGs with non-negative-valued local functions, a popular approach is to approximate  $Z(N)$  by the Bethe partition sum  $Z_B(N)$ , a quantity which is defined via the minimum of the Bethe free energy function [6]. A reason for the popularity of the Bethe approximation is that in many cases it can be found efficiently with the help of the sum-product algorithm [1], [3], [6].

In contrast to the above, analytical definition of  $Z_B(N)$ , it was shown in [7] that  $Z_B(N)$  admits the following, combinatorial characterization in terms of graph covers. Namely,

$$Z_B(N) = \limsup_{M \rightarrow \infty} Z_{B,M}(N), \quad (2)$$

$$Z_{B,M}(N) \triangleq \sqrt[M]{\left\langle Z(\tilde{N}) \right\rangle_{\tilde{N} \in \tilde{\mathcal{N}}_M}}. \quad (3)$$

Here the expression under the root sign represents the (arithmetic) average of  $Z(\tilde{N})$  over all  $M$ -covers  $\tilde{N}$  of  $N$ ,  $M \geq 1$ .

Note that we can write

$$\underbrace{\frac{Z(N)}{Z_B(N)}}_{\textcircled{1}} = \underbrace{\frac{Z(N)}{Z_{B,2}(N)}}_{\textcircled{2}} \cdot \underbrace{\frac{Z_{B,2}(N)}{Z_B(N)}}_{\textcircled{3}}. \quad (4)$$

For many NFGs, a significant contribution to the ratio  $\textcircled{1}$  comes from the ratio  $\textcircled{2}$ . Therefore, understanding the ratio  $\textcircled{2}$  can give useful insights to understanding the ratio  $\textcircled{1}$ .

The aim of the present paper is to develop techniques towards better understanding and quantifying the ratio  $\textcircled{2}$ . In particular, we will study the partition sum of double covers of log-supermodular NFGs and thereby give an alternative proof for a special case of a theorem by Ruzozzi [8].

On the one hand, the contributions here can be seen as adding another tool in the holographic transformations toolbox for NFGs [9], [10], and, on the other hand, they can be seen as adding another tool to the toolbox for relating the partition sum and its Bethe approximation (see, e.g., [4], [5], [11], [12]).

## A. Overview

The paper is structured as follows. In Section II we give a brief introduction to NFGs and their double covers. In Section III we present a novel technique for analyzing double covers. In Section IV we apply this technique to the analysis of a special class of log-supermodular NFGs. Finally, in Section V we conclude the paper.

## II. NORMAL FACTOR GRAPHS AND THEIR FINITE COVERS

Factor graphs are a convenient way to represent multivariate functions [1]. In this paper we use a variant called normal factor graphs (NFGs) [2] (also called Forney-style factor graphs [3]), where variables are associated with edges. The following example is taken from [7].

**Example 1** Consider the multivariate function

$$g(a_{e_1}, \dots, a_{e_8}) \triangleq f_1(a_{e_1}, a_{e_2}, a_{e_5}) \cdot f_2(a_{e_2}, a_{e_3}, a_{e_6}) \cdot f_3(a_{e_3}, a_{e_4}, a_{e_7}) \cdot f_4(a_{e_5}, a_{e_6}, a_{e_8}) \cdot f_5(a_{e_7}, a_{e_8}),$$

where the so-called global function  $g$  is the product of the so-called local functions  $f_1, f_2, f_3, f_4$ , and  $f_5$ . The decomposition of this global function as a product of local functions can be depicted with the help of an NFG  $N$  as shown in Fig. 1. In particular, the NFG  $N$  consists of

- the function nodes  $f_1, f_2, f_3, f_4$ , and  $f_5$ ;
- the half edges  $e_1$  and  $e_4$  (sometimes also called “external edges”);
- the full edges  $e_2, e_3, e_5, e_6, e_7$ , and  $e_8$  (sometimes also called “internal edges”).

In general,

- a function node  $f$  represents the local function  $f$ ;
- with an edge  $e$  we associate the variable  $A_e$  (note that a realization of the variable  $A_e$  is denoted by  $a_e$ );

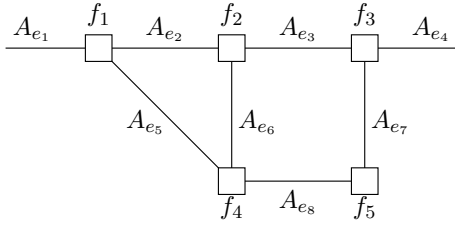


Fig. 1. NFG N used in Example 1.

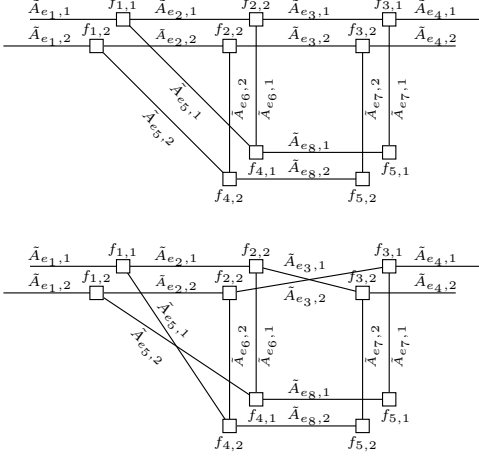


Fig. 2. Two possible 2-covers of the NFG N that is shown in Fig. 1.

- an edge  $e$  is incident on a function node  $f$  if and only if  $a_e$  appears as an argument of the local function  $f$ .

Finally, we associate with  $N$  the partition sum  $Z(N)$  as defined in (1). (Note that we do not consider any temperature dependence of  $Z(N)$  in this paper.)

Throughout this paper, we will essentially use the same notation as in [7]. The only exceptions are  $Z(N)$  instead of  $Z_G(N)$  for the partition sum, and  $f$  instead of  $g_f$  for local functions. (For notations which are not defined in this paper, we refer the reader to Sections II and IV of [7].) Note that for the rest of this paper, we assume that local functions in the base NFG  $N$  take on only non-negative real values, i.e.,  $f(a_f) \in \mathbb{R}_{\geq 0}$  for all  $f$  and all  $a_f$ .

Central to this paper are also finite graph covers of an NFG. (For the definition of a finite graph cover, see, e.g., [7].) The following example is taken from [7].

**Example 2** Consider again the NFG  $N$  that is discussed in Example 1 and depicted in Fig. 1. Two possible 2-covers of this (base) NFG are shown in Fig. 2. The first graph cover is “trivial” in the sense that it consists of two disjoint copies of the NFG in Fig. 1. The second graph cover is “more interesting” in the sense that the edge permutations are such that the two copies of the base NFG are intertwined. (Of course, both graph covers are equally valid.)

Based on finite graph covers, one can define the degree- $M$  Bethe partition sum  $Z_{B,M}(N)$  as in (3) for any  $M \geq 1$ .

$$\begin{array}{c} Z_{B,M}(N)|_{M \rightarrow \infty} = Z_B(N) \\ \downarrow \\ Z_{B,M}(N) \\ \downarrow \\ Z_{B,M}(N)|_{M=1} = Z(N) \end{array}$$

 Fig. 3. The degree- $M$  Bethe partition function of the NFG  $N$  for different values of  $M$ .

With this, one can prove the alternative expression for  $Z_B(N)$  in (2). When considering the value of  $Z_{B,M}(N)$  from  $M = 1$  to  $M = \infty$ , one goes from  $Z(N)$  to  $Z_B(N)$  as shown in Fig. 3.

In this paper, we also need the definition of a binary log-supermodular NFG: it is an NFG with binary variables and log-supermodular local functions. Recall that a local function  $f : \{0, 1\}^{d_f} \rightarrow \mathbb{R}_{\geq 0}$  is called log-supermodular if

$$f(a'_f) \cdot f(a''_f) \leq f(a'_f \wedge a''_f) \cdot f(a'_f \vee a''_f)$$

holds for all  $a'_f, a''_f \in \{0, 1\}^{d_f}$ , where

$$\begin{aligned} (a'_f \wedge a''_f)_e &\triangleq \min(a'_{f,e}, a''_{f,e}), \quad e \in \mathcal{E}_f, \\ (a'_f \vee a''_f)_e &\triangleq \max(a'_{f,e}, a''_{f,e}), \quad e \in \mathcal{E}_f. \end{aligned}$$

Similarly,  $f : \{0, 1\}^{d_f} \rightarrow \mathbb{R}_{\geq 0}$  is called log-submodular if

$$f(a'_f) \cdot f(a''_f) \geq f(a'_f \wedge a''_f) \cdot f(a'_f \vee a''_f)$$

holds for all  $a'_f, a''_f \in \{0, 1\}^{d_f}$ .

With a function like  $f : \{0, 1\}^2 \rightarrow \mathbb{R}$ , it is natural to associate the matrix

$$\mathbf{T}_f \triangleq \begin{pmatrix} f(0, 0) & f(0, 1) \\ f(1, 0) & f(1, 1) \end{pmatrix}.$$

Note that the determinant of  $\mathbf{T}_f$  is

$$\det(\mathbf{T}_f) = f(0, 0) \cdot f(1, 1) - f(1, 0) \cdot f(0, 1).$$

Clearly,

$$\begin{aligned} \text{if } f \text{ is log-supermodular then } \det(\mathbf{T}_f) &\geq 0; \\ \text{if } f \text{ is log-submodular then } \det(\mathbf{T}_f) &\leq 0. \end{aligned}$$

The following theorem was shown by Ruozzi [8]. Its elegant proof was based on the four-function theorem and generalizations thereof.

**Theorem 3 ([8])** Let  $N$  be a binary log-supermodular NFG. Then for any  $M$ -cover  $\tilde{N}$  of  $N$ ,  $M \geq 1$ , it holds that

$$Z(\tilde{N}) \leq Z(N)^M. \quad (5)$$

Combining (5) with (3), one obtains  $Z_{B,M}(N) \leq Z(N)$  for all  $M \geq 1$ . Moreover, using (2), one obtains  $Z_B(N) \leq Z(N)$ . Note that before Ruozzi’s paper, the result  $Z_B(N) \leq Z(N)$  had been proven by Sudderth *et al.* [13] for some special cases of binary log-supermodular graphical models. After Ruozzi’s paper, Weller and Jebara [14] gave an alternative proof for binary log-supermodular NFGs where all function nodes (except the equality function nodes) have degree two.

### III. ANALYZING DOUBLE COVERS

Consider an arbitrary NFG  $N$  without half edges,<sup>1</sup> where  $\mathcal{A}_e \triangleq \{0, 1\}$  for all edges  $e \in \mathcal{E}$ . In this section we present a novel approach for analyzing  $Z(\tilde{N})$  for some double cover  $\tilde{N}$  of  $N$ , ultimately towards comparing  $Z_{B,2}(N)$  with  $Z(N)$  and  $Z_B(N)$ . This approach consists of two steps:

- In the first step, we associate a new NFG with  $\tilde{N}$ . We will call it the merged double cover NFG (MDC-NFG) associated with  $\tilde{N}$  and denote it by  $\tilde{N}_{\text{MDC}}$ .
- In the second step, we apply a suitable holographic transform [9], [10] to the MDC-NFG. The resulting NFG is called the transformed MDC-NFG and denoted by  $\hat{N}_{\text{MDC}}$ . The key property of  $\tilde{N}_{\text{MDC}}$  and  $\hat{N}_{\text{MDC}}$  is

$$Z(\tilde{N}) = Z(\tilde{N}_{\text{MDC}}) = Z(\hat{N}_{\text{MDC}}). \quad (6)$$

The proposed approach is visualized in Fig. 4 with the help of an example NFG  $N$ .

- Fig. 4(a) shows a part of a larger NFG  $N$ . Here,  $f_1$  and  $f_2$  are function nodes of degree three.
- Figs. 4(b) and 4(c) show the same part as in Fig. 4(a) for different double covers of  $N$ .
- Starting with a given double cover  $\tilde{N}$  of  $N$ , the associated MDC-NFG  $\tilde{N}_{\text{MDC}}$  in Fig. 4(d) is obtained as follows.
  - For every function node  $f_j$  in  $N$  we close-the-box (see [3], [15]) around every pair of function nodes  $f_{j,1}$  and  $f_{j,2}$  in  $\tilde{N}$  associated with  $f_j$  and call the resulting function  $\tilde{f}_j$ . Because there are no variables to be summed over,  $\tilde{f}_j$  is simply the product of  $f_{j,1}$  and  $f_{j,2}$ . Note that if the function  $f_j$  has  $d_j$  arguments, i.e.,  $f_j : \{0, 1\}^{d_j} \rightarrow \mathbb{R}$ , then

$$\tilde{f}_j : \{(0, 0), (0, 1), (1, 0), (1, 1)\}^{d_j} \rightarrow \mathbb{R}.$$

- For every edge  $e$  in  $N$ , we introduce the local function  $\tilde{E}_e$  which encodes the non-crossing / the crossing of the pair of edges in  $\tilde{N}$  associated with  $e$ . The local function  $\tilde{E}_e$  is defined such that  $\tilde{a}_{e,s} = 0$  corresponds to the case where there is no crossing of the pair of edges in  $\tilde{N}$  and  $\tilde{a}_{e,s} = 1$  corresponds to the case where there is a crossing of the pair of edges in  $\tilde{N}$ . With this, the matrices associated with

$$\begin{aligned} \tilde{E}_e((\tilde{a}_{f_1,e,1}, \tilde{a}_{f_1,e,2}), (\tilde{a}_{f_2,e,1}, \tilde{a}_{f_2,e,2}), \tilde{a}_{e,s}=0) , \\ \tilde{E}_e((\tilde{a}_{f_1,e,1}, \tilde{a}_{f_1,e,2}), (\tilde{a}_{f_2,e,1}, \tilde{a}_{f_2,e,2}), \tilde{a}_{e,s}=1) \end{aligned}$$

are, respectively,

$$\tilde{E}_{\text{nocross}} \triangleq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{E}_{\text{cross}} \triangleq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(The “s” in  $a_{e,s}$  stands for “switch.”)

- We define

$$\begin{aligned} \tilde{f}_{e,s}(0) &\triangleq 1, & \tilde{f}_{e,s}(1) &\triangleq 0 & (\text{no crossing}), \\ \tilde{f}_{e,s}(0) &\triangleq 0, & \tilde{f}_{e,s}(1) &\triangleq 1 & (\text{crossing}). \end{aligned}$$

<sup>1</sup>Because we are mainly interested in the partition sum of  $N$  and because summing over variables associated with half edges is straightforward, considering only NFGs without half edges is no major restriction.

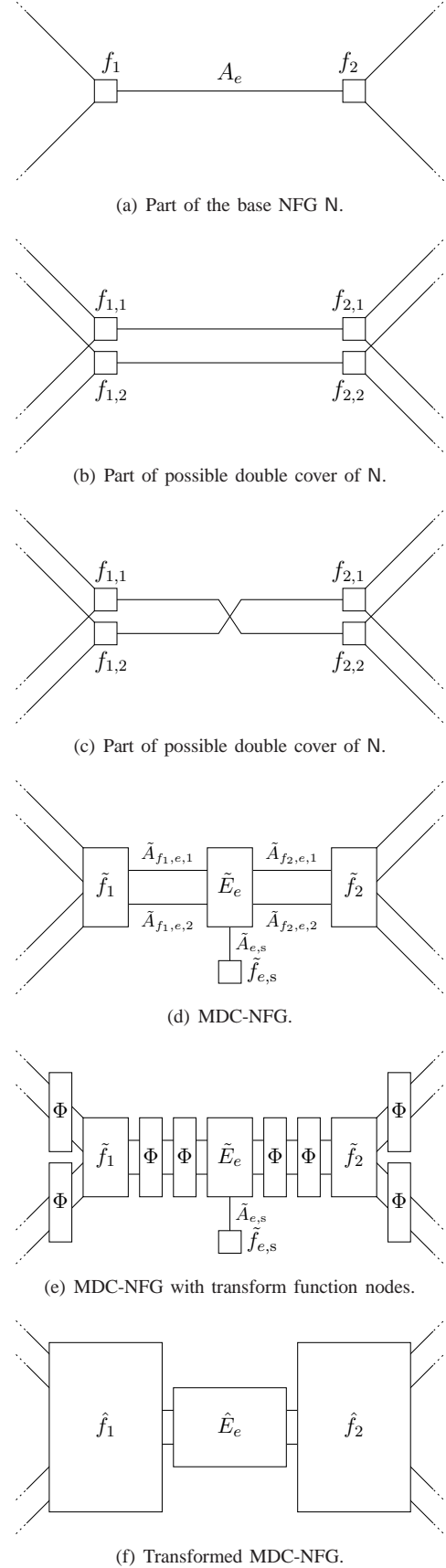


Fig. 4. Partial NFGs exemplifying the analysis technique in Section III.

- One can verify that there is a bijection between valid configurations in  $\tilde{N}$  and valid configurations in  $\tilde{N}_{\text{MDC}}$ , along with their corresponding global function values being equal. Therefore,  $Z(\tilde{N}_{\text{MDC}}) = Z(\tilde{N})$ .
- The NFG in Fig. 4(f) is obtained from the NFG in Fig. 4(d) by introducing multiple instances of the function node  $\Phi$  via opening-the-box. Here, the local function

$$\Phi : \{(0,0), (0,1), (1,0), (1,1)\}^2 \rightarrow \mathbb{R}$$

is specified via the matrix associated with  $\Phi$ , namely,

$$T_\Phi \triangleq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that  $T_\Phi^\top = T_\Phi$  and  $T_\Phi^{-1} = T_\Phi$ .

- Finally, the transformed MDC-NFG  $\hat{N}_{\text{MDC}}$  in Fig. 4(f) is obtained from Fig. 4(e) by applying several closing-the-box operations. Namely, for every edge  $e \in E$ , the function node  $\hat{E}_e$  is obtained by closing-the-box around  $\tilde{E}_e$ , the two adjacent  $\Phi$ -function nodes, and the  $\tilde{f}_{e,s}$  function node. With this, if the pair of edges in  $\tilde{N}$  corresponding to  $e$  does not cross / does cross then the matrix associated with  $\hat{E}_e$  equals, respectively,

$$\hat{E}_{\text{nocross}} \triangleq T_\Phi \cdot \tilde{E}_{\text{nocross}} \cdot T_\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\hat{E}_{\text{cross}} \triangleq T_\Phi \cdot \tilde{E}_{\text{cross}} \cdot T_\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For every  $f_j \in \mathcal{F}$ , the function node  $\hat{f}_j$  is obtained by closing-the-box around the function node  $\tilde{f}_j$  and the  $d_j$  adjacent  $\Phi$ -function nodes, where  $d_j$  is the degree of the function node  $f_j$ . The above construction implies (see [9], [10]) that  $Z(\hat{N}_{\text{MDC}}) = Z(\tilde{N}_{\text{MDC}})$ . Combining this with the equality  $Z(\tilde{N}_{\text{MDC}}) = Z(\tilde{N})$ , we obtain (6).

Let us conclude this section by considering a variation of the definition of  $\tilde{f}_{e,s}$  and with that a variation of the definition of  $\hat{N}_{\text{MDC}}$ . Namely, for every  $e \in \mathcal{E}$ , define  $\tilde{f}_{e,s}(0) \triangleq \frac{1}{2}$  and  $\tilde{f}_{e,s}(1) \triangleq \frac{1}{2}$ . Then the matrix associated with  $\hat{E}_e$  equals

$$T_{\hat{E}_e} \triangleq \frac{1}{2} \cdot \hat{E}_{\text{nocross}} + \frac{1}{2} \cdot \hat{E}_{\text{cross}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

These definitions allow one to formulate the following theorem, whose proof we omit.

**Theorem 4** *For the NFG  $\hat{N}_{\text{MDC}}$  as just specified, it holds that*

$$Z_{B,2}(N) = \sqrt{Z(\hat{N}_{\text{MDC}})}. \quad (7)$$

Note that, in contrast to (3), only a single NFG appears in the expression on the right-hand side of (7).

#### IV. LOG-SUPERMODULAR NFGs

In this section we apply the technique from Section III to analyze the following class of NFGs: it consists of all NFGs without half edges and

- where  $\mathcal{A}_e = \{0,1\}$  for all edges  $e \in \mathcal{E}$ ,
- where all local functions are log-supermodular, and
- where all function nodes have degree 2 or 3, except for equality indicator function nodes which may have arbitrary degree at least 2.<sup>2</sup>

**Theorem 5** *Let  $N$  be an NFG from the class of NFGs specified above let  $\tilde{N}$  be an arbitrary double cover of  $N$ . Then*

$$Z(\tilde{N}) \leq Z(N)^2.$$

*Proof:* (sketch) Let  $\hat{N}_{\text{MDC}}$  and  $\hat{N}_{\text{MDC}}^{\text{trivial}}$  be associated with the double cover  $\tilde{N}$  and the trivial double cover  $\tilde{N}^{\text{trivial}}$ , respectively. Their partition sums are  $Z(\hat{N}_{\text{MDC}}) = \sum_{\hat{\mathbf{a}}} \hat{g}(\hat{\mathbf{a}})$  and  $Z(\hat{N}_{\text{MDC}}^{\text{trivial}}) = \sum_{\hat{\mathbf{a}}} \hat{g}^{\text{trivial}}(\hat{\mathbf{a}})$ , respectively. Note that both sums are over the same set of configurations. From the results in Sections III and the upcoming results in Section IV it follows that  $\hat{g}^{\text{trivial}}(\hat{\mathbf{a}}) \geq 0$  and  $\hat{g}(\hat{\mathbf{a}}) = \pm \hat{g}^{\text{trivial}}(\hat{\mathbf{a}})$  for all  $\hat{\mathbf{a}}$ , which implies  $Z(\hat{N}_{\text{MDC}}) \leq Z(\hat{N}_{\text{MDC}}^{\text{trivial}})$ . Finally, because  $Z(\hat{N}_{\text{MDC}}) = Z(\tilde{N})$  and  $Z(\hat{N}_{\text{MDC}}^{\text{trivial}}) = Z(\tilde{N}^{\text{trivial}}) = Z(N)^2$ , we obtain the promised result. ■

##### A. Arbitrary Log-Supermodular Function Node of Degree 2

Let  $f$  be a log-supermodular function with two arguments; let  $t_{00} \triangleq f(0,0)$ ,  $t_{01} \triangleq f(0,1)$ ,  $t_{10} \triangleq f(1,0)$ ,  $t_{11} \triangleq f(1,1)$ . With this, the matrices associated with  $f$ ,  $\tilde{f}$ , and  $\hat{f}$  are, respectively,

$$T_f \triangleq \begin{pmatrix} t_{00} & t_{01} \\ t_{10} & t_{11} \end{pmatrix}, \quad T_{\tilde{f}} \triangleq \begin{pmatrix} t_{00}t_{00} & t_{00}t_{01} & t_{01}t_{00} & t_{01}t_{01} \\ t_{00}t_{10} & t_{00}t_{11} & t_{01}t_{10} & t_{01}t_{11} \\ t_{10}t_{00} & t_{10}t_{01} & t_{11}t_{00} & t_{11}t_{01} \\ t_{10}t_{10} & t_{10}t_{11} & t_{11}t_{10} & t_{11}t_{11} \end{pmatrix},$$

$$T_{\hat{f}} \triangleq T_\Phi \cdot T_{\tilde{f}} \cdot T_\Phi = \begin{pmatrix} t_{00}t_{00} & \sqrt{2}t_{00}t_{01} & 0 & t_{01}t_{01} \\ \sqrt{2}t_{00}t_{10} & \text{perm}(T_f) & 0 & \sqrt{2}t_{01}t_{11} \\ 0 & 0 & \det(T_f) & 0 \\ t_{10}t_{10} & \sqrt{2}t_{10}t_{11} & 0 & t_{11}t_{11} \end{pmatrix},$$

where  $\text{perm}(T_f) \triangleq t_{00}t_{11} + t_{10}t_{01}$ . Because  $f$  is log-supermodular,  $\det(T_f)$  is non-negative, and so all entries of  $T_{\hat{f}}$  are non-negative.

##### B. Arbitrary Log-Supermodular Function Node of Degree 3

Let  $f(a_1, a_2, a_3)$  be a log-supermodular function with three arguments and let  $t_{000} \triangleq f(0,0,0)$ ,  $t_{001} \triangleq f(0,0,1)$ , etc. Moreover, let  $T_{f|a_3=0}$  and  $T_{f|a_3=1}$  be the matrices associated with the functions  $f(a_1, a_2, 0)$  and  $f(a_1, a_2, 1)$ , respectively. (Clearly, if  $f(a_1, a_2, a_3)$  is a log-supermodular function, then

<sup>2</sup>Note that equality indicator functions are log-supermodular.



also  $f(a_1, a_2, 0)$  and  $f(a_1, a_2, 1)$  are log-supermodular functions.) The matrices  $\mathbf{T}_{f|a_1=0}$ ,  $\mathbf{T}_{f|a_1=1}$ ,  $\mathbf{T}_{f|a_2=0}$ , and  $\mathbf{T}_{f|a_2=1}$  are defined analogously. Then the  $4 \times 4 \times 4$  array  $\mathbf{T}_f$  associated with  $\hat{f}$  is given by

$$\begin{pmatrix} t_{000}t_{000} & \sqrt{2}t_{000}t_{010} & 0 & t_{010}t_{010} \\ \sqrt{2}t_{000}t_{100} & \text{perm}(\mathbf{T}_{f|a_3=0}) & 0 & \sqrt{2}t_{010}t_{110} \\ 0 & 0 & \det(\mathbf{T}_{f|a_3=0}) & 0 \\ t_{100}t_{100} & \sqrt{2}t_{100}t_{110} & 0 & t_{110}t_{110} \end{pmatrix},$$

$$\begin{pmatrix} \sqrt{2}t_{000}t_{001} & \text{perm}(\mathbf{T}_{f|a_1=0}) & 0 & \sqrt{2}t_{010}t_{011} \\ \text{perm}(\mathbf{T}_{f|a_2=0}) & \hat{f}(\hat{0}, \hat{0}, \hat{0}) & 0 & \text{perm}(\mathbf{T}_{f|a_2=1}) \\ 0 & 0 & \hat{f}(\hat{1}, \hat{1}, \hat{0}) & 0 \\ \sqrt{2}t_{100}t_{101} & \text{perm}(\mathbf{T}_{f|a_1=1}) & 0 & \sqrt{2}t_{110}t_{111} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & \det(\mathbf{T}_{f|a_1=0}) & 0 \\ 0 & 0 & \hat{f}(\hat{0}, \hat{1}, \hat{1}) & 0 \\ \det(\mathbf{T}_{f|a_1=0}) & \hat{f}(\hat{1}, \hat{0}, \hat{1}) & 0 & \det(\mathbf{T}_{f|a_1=0}) \\ 0 & 0 & \det(\mathbf{T}_{f|a_1=1}) & 0 \end{pmatrix},$$

$$\begin{pmatrix} t_{001}t_{001} & \sqrt{2}t_{001}t_{011} & 0 & t_{011}t_{011} \\ \sqrt{2}t_{001}t_{101} & \text{perm}(\mathbf{T}_{f|a_3=1}) & 0 & \sqrt{2}t_{011}t_{111} \\ 0 & 0 & \det(\mathbf{T}_{f|a_3=1}) & 0 \\ t_{101}t_{101} & \sqrt{2}t_{101}t_{111} & 0 & t_{111}t_{111} \end{pmatrix},$$

where

$$\begin{aligned} \hat{f}(\hat{0}, \hat{0}, \hat{0}) &= \gamma \cdot (t_{000}t_{111} + t_{100}t_{011} + t_{010}t_{101} + t_{000}t_{110}), \\ \hat{f}(\hat{1}, \hat{0}, \hat{1}) &= \gamma \cdot (t_{000}t_{111} - t_{100}t_{011} + t_{010}t_{101} - t_{001}t_{110}), \\ \hat{f}(\hat{0}, \hat{1}, \hat{1}) &= \gamma \cdot (t_{000}t_{111} + t_{100}t_{011} - t_{010}t_{101} - t_{001}t_{110}), \\ \hat{f}(\hat{1}, \hat{1}, \hat{0}) &= \gamma \cdot (t_{000}t_{111} - t_{100}t_{011} - t_{010}t_{101} + t_{001}t_{110}), \end{aligned}$$

and where  $\hat{0} \triangleq (0, 1)$ ,  $\hat{1} \triangleq (1, 0)$ , and  $\gamma \triangleq 1/\sqrt{2}$ .

**Lemma 6** All entries of  $\mathbf{T}_f$  are non-negative.

*Proof:* For most entries of  $\mathbf{T}_f$  the statement is clearly true. Moreover, the log-supermodularity of  $f$  implies that all entries based on determinants must be non-negative. Also, from the definition of  $\hat{f}(\hat{0}, \hat{0}, \hat{0})$ , it follows that  $\hat{f}(\hat{0}, \hat{0}, \hat{0}) \geq 0$ . It only remains to show  $\hat{f}(\hat{1}, \hat{0}, \hat{1}) \geq 0$ ,  $\hat{f}(\hat{0}, \hat{1}, \hat{1}) \geq 0$ , and  $\hat{f}(\hat{1}, \hat{1}, \hat{0}) \geq 0$ . In this proof we show  $\hat{f}(\hat{0}, \hat{1}, \hat{1}) \geq 0$ . Analogous lines of reasoning yield  $\hat{f}(\hat{1}, \hat{0}, \hat{1}) \geq 0$  and  $\hat{f}(\hat{1}, \hat{1}, \hat{0}) \geq 0$ .

Let  $s_0 \triangleq \gamma \cdot t_{000}t_{111}$ ,  $s_1 \triangleq \gamma \cdot t_{100}t_{011}$ ,  $s_2 \triangleq \gamma \cdot t_{010}t_{101}$ ,  $s_3 \triangleq \gamma \cdot t_{001}t_{110}$ . From log-supermodularity of  $f$  it follows that  $s_0 \geq s_1$ ,  $s_0 \geq s_2$ ,  $s_0 \geq s_3$ , and  $s_0s_1 \geq s_2s_3$ .

The inequality  $\hat{f}(\hat{0}, \hat{1}, \hat{1}) \geq 0$  is equivalent to the inequality  $s_0 + s_1 - s_2 - s_3 \geq 0$ . We show the latter inequality by considering two cases:  $0 \leq s_2 \leq s_1 \leq s_0$  and  $0 \leq s_1 < s_2 \leq s_0$ .

- Assume  $0 \leq s_2 \leq s_1 \leq s_0$ . Then  $s_0 + s_1 - s_2 - s_3 \geq 0$  follows immediately from the combination of  $s_0 \geq s_3$  and  $s_1 \geq s_2$ .
- Assume  $0 \leq s_1 < s_2 \leq s_0$ . Then  $(s_0 - s_2) \cdot (s_2 - s_1) \geq 0$  implies  $s_0s_2 + s_1s_2 - s_2^2 - s_0s_1 \geq 0$ . Using  $s_0s_1 \geq s_2s_3$ , this inequality implies  $s_0s_2 + s_1s_2 - s_2^2 - s_2s_3 \geq 0$ , which in turn implies  $s_0 + s_1 - s_2 - s_3 \geq 0$  because  $s_2 > 0$ . ■

### C. Equal Function Node of Arbitrary Degree At Least 2

We have the following theorem, whose proof is omitted.

**Theorem 7** Let  $f$  be an equality indicator function with  $d \geq 2$  arguments. Then  $\hat{f}((\hat{a}_{1,1}, \hat{a}_{1,2}), \dots, (\hat{a}_{d,1}, \hat{a}_{d,2}))$  equals

$$\begin{cases} 1 & \text{if } \hat{a}_{i,m} = 0 \ \forall i \in \{1, \dots, d\}, \ m \in \{1, 2\} \\ 1 & \text{if } \hat{a}_{i,m} = 1 \ \forall i \in \{1, \dots, d\}, \ m \in \{1, 2\} \\ 2^{1-d/2} & \text{if } (\hat{a}_{i,1}, \hat{a}_{i,2}) \in \{(0, 1), (1, 0)\} \ \forall i \in \{1, \dots, d\} \\ & \text{and } \sum_{i=1}^d \hat{a}_{i,1} = 0 \pmod{2} \\ 0 & \text{otherwise} \end{cases}$$

### V. CONCLUSION AND OUTLOOK

We leave it as an open problem to generalize Theorem 5 to all binary log-supermodular NFGs, *i.e.*, to the setup of Theorem 3. Moreover, we will discuss elsewhere how the results in Sections III and IV can be used to quantify the ratio ② in (4).

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### REFERENCES

- [1] F. R. Kschischang, B. J. Frey, and H.-A. Loeliger, "Factor graphs and the sum-product algorithm," *IEEE Trans. Inf. Theory*, vol. 47, no. 2, pp. 498–519, Feb. 2001.
- [2] G. D. Forney, Jr., "Codes on graphs: normal realizations," *IEEE Trans. Inf. Theory*, vol. 47, no. 2, pp. 520–548, Feb. 2001.
- [3] H.-A. Loeliger, "An introduction to factor graphs," *IEEE Sig. Proc. Mag.*, vol. 21, no. 1, pp. 28–41, Jan. 2004.
- [4] F. Parvaresh and P. O. Vontobel, "Approximately counting the number of constrained arrays via the sum-product algorithm," in *Proc. IEEE Int. Symp. Inf. Theory*, Cambridge, MA, USA, Jul. 1–6 2012, pp. 279–283.
- [5] P. O. Vontobel, "Counting balanced sequences w/o forbidden patterns via the Bethe approximation and loop calculus," in *Proc. IEEE Int. Symp. Inf. Theory*, Honolulu, HI, USA, Jun. 29–Jul. 4 2014, pp. 1608–1612.
- [6] J. S. Yedidia, W. T. Freeman, and Y. Weiss, "Constructing free-energy approximations and generalized belief propagation algorithms," *IEEE Trans. Inf. Theory*, vol. 51, no. 7, pp. 2282–2312, Jul. 2005.
- [7] P. O. Vontobel, "Counting in graph covers: a combinatorial characterization of the Bethe entropy function," *IEEE Trans. Inf. Theory*, vol. 59, no. 9, pp. 6018–6048, Sep. 2013.
- [8] N. Ruozzi, "The Bethe partition function of log-supermodular graphical models," in *Proc. Neural Inf. Proc. Sys. Conf.*, Lake Tahoe, NV, USA, Dec. 3–6 2012.
- [9] A. Al-Bashabsheh and Y. Mao, "Normal factor graphs and holographic transformations," *IEEE Trans. Inf. Theory*, vol. 57, no. 2, pp. 752–763, Feb. 2011.
- [10] G. D. Forney, Jr. and P. O. Vontobel, "Partition functions of normal factor graphs," in *Proc. Inf. Theory Appl. Workshop*, UC San Diego, La Jolla, CA, USA, Feb. 6–11 2011.
- [11] M. Chertkov and V. Y. Chernyak, "Loop series for discrete statistical models on graphs," *J. Stat. Mech.: Theory and Experiment*, p. P06009, Jun. 2006.
- [12] R. Mori, "Loop calculus for nonbinary alphabets using concepts from information geometry," *IEEE Trans. Inf. Theory*, vol. 61, no. 4, pp. 1887–1904, Apr. 2015.
- [13] E. B. Sudderth, M. J. Wainwright, and A. S. Willsky, "Loop series and Bethe variational bounds in attractive graphical models," in *Proc. Neural Inf. Proc. Sys. Conf.*, Vancouver, Canada, Dec. 3–8 2007.
- [14] A. Weller and T. Jebara, "Clamping variables and approximate inference," in *Proc. Neural Inf. Proc. Sys. Conf.*, Montreal, Canada, Dec. 8–13 2014, pp. 909–917.
- [15] P. O. Vontobel and H.-A. Loeliger, "On factor graphs and electrical networks," in *Mathematical Systems Theory in Biology, Communication, Computation, and Finance, IMA Volumes in Math. & Appl.*, D. Gilliam and J. Rosenthal, Eds. Springer Verlag, 2003.